TEMPERATURE FIELD OF A PLATE WITH A DISCRETE ENERGY SOURCE

G. N. Dul'nev and B. V. Pol'shchikov

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An approximate solution of the problem of stationary temperature field of a plate with a discrete energy source is presented. The results of the approximate and exact solutions are compared.

A rectangular plate of thickness δ and dimensions L_x, L_y has a region U of rectangular form and dimensions $2\Delta\xi$, $2\Delta\eta$; heat sources with a power P are uniformly distributed inside this region (Fig. 1). Heat exchange from the ends of the plate is absent, while from the two main surfaces heat through convection and radiation is dissipated into the surrounding medium. The coefficients of heat exchange do not depend on the temperature, and are equal to α_1, α_2. The stationary temperature field of such a plate is described by the following differential equation and the boundary conditions [1]:

$$\frac{\partial^2\vartheta}{\partial x^2}+\frac{\partial^2\vartheta}{\partial y^2}-a^2\vartheta=-g_s 1\{U\},\ \vartheta=t-t_c, \tag{1}$$

$$g_s=\frac{p}{4\Delta\xi\Delta\eta\delta\lambda},\quad a=\frac{\alpha_1+\alpha_2}{\lambda\delta},\quad 1\{U\}=\begin{cases}1 \text{ — inside the region}\\ 0 \text{ — inside the region}\end{cases}$$

$$\left.\frac{\partial\vartheta}{\partial i}\right|_{i=0}=\left.\frac{\partial\vartheta}{\partial i}\right|_{i=L_i}=0\quad (i=x,\ y). \tag{2}$$

We introduce the following dimensionless quantities:

$$\bar{i}=\frac{i}{L_i},\quad \varepsilon_i=\left(\frac{\delta}{L_i}\right)^2,\quad b^2=\frac{\delta(\alpha_1+\alpha_2)}{\lambda},$$
$$N=\frac{4\Delta\xi\Delta\eta\vartheta\lambda}{P\delta},\quad \bar{\xi}=\frac{\xi}{L_x},\quad \bar{\eta}=\frac{\eta}{L_y}, \tag{3}$$
$$\Delta\bar{\xi}=\frac{\Delta\xi}{L_x},\ \Delta\bar{\eta}=\frac{\Delta\eta}{L_y}$$

and write the system (1), (2) in the form

$$\varepsilon_x\frac{\partial^2 N}{\partial\bar{x}^2}+\varepsilon_y\frac{\partial^2 N}{\partial\bar{y}^2}-b^2N=-1\{U\}, \tag{4}$$

$$\left.\frac{\partial N}{\partial\bar{i}}\right|_{\bar{i}=0}=\left.\frac{\partial N}{\partial\bar{i}}\right|_{\bar{i}=L_i}=0. \tag{5}$$

An exact analytical solution of the system (4), (5) was obtained by the finite integral transformation method and is presented in [1]. This solution is very complicated, and the series slowly converge, especially in the region of a local source. Approximate solutions of this problem, found in [1, 2], are also fairly cumbersome and inconvenient for practical calculations. Below we give a solution of the problem thus formulated by means of the generalized Kantorovich method whose essential features are presented in [3].

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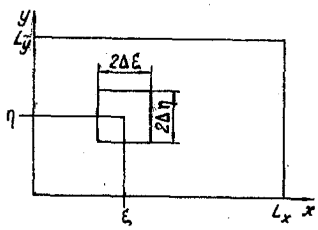

Fig. 1. Rectangular plate with a heat source.

In accordance with [3] we apply, term by term, the averaging operator I_x to Eq. (4):

$$I_x(f)=\int_0^1 N d\bar{y} = \langle N_x \rangle,$$

$$I_x\left(\varepsilon_x \frac{\partial^2 N}{\partial \bar{x}^2}\right)=\varepsilon_x \frac{d^2\langle N_x\rangle}{d\bar{x}^2},\quad I_x\left(\varepsilon_y \frac{\partial^2 N}{\partial \bar{y}^2}\right)=\varepsilon_y\left[\frac{\partial N}{\partial \bar{y}}\right]_{\bar{y}=0}^{\bar{y}=1}.$$

On the basis of the boundary conditions (5)

$$\left[\frac{\partial N}{\partial \bar{y}}\right]_{\bar{y}=0}^{\bar{y}=1}=0,$$

consequently,

$$I_x\left(\varepsilon_y \frac{\partial^2 N}{\partial \bar{y}^2}\right)=0.$$

Analogously, we obtain

$$I_x(b^2N)=b^2\langle N_x\rangle,\quad I_x[1\{U\}]=2\Delta\bar{\eta}1\{\bar{x}\},$$

where

$$1\{\bar{x}\}=\begin{cases}1 & \text{for} \quad \bar{x}\in(\bar{\xi}-\Delta\bar{\xi};\ \bar{\xi}+\Delta\bar{\xi})\\ 0 & \text{for} \quad \bar{x}\notin(\bar{\xi}-\Delta\bar{\xi};\ \bar{\xi}+\Delta\bar{\xi})\end{cases}$$

Uniting the results of the term by term effect of the operator I_x, we reduce Eq. (4) to the ordinary differential equation relative to $\langle N_x\rangle$:

$$\frac{d^2\langle N_x\rangle}{d\bar{x}^2}-p_x^2\langle N_x\rangle=-\frac{2\Delta\bar{\eta}}{\varepsilon_x}1\{\bar{x}\},\tag{6}$$

$$p_x^2=b^2/\varepsilon_x.\tag{7}$$

An application of the operator I_x to the boundary conditions (5) for $\bar{x}=0$ leads to the following condition for $\langle N_x\rangle$:

$$\left.\frac{d\langle N_x\rangle}{d\bar{x}}\right|_{\bar{x}=0}=\left.\frac{d\langle N_x\rangle}{d\bar{x}}\right|_{\bar{x}=1}=0.\tag{8}$$

Integrating Eq. (6) and satisfying the boundary conditions (8), we obtain

$$\langle N_x \rangle = \frac{2\Delta\bar{\eta}}{\varepsilon_x p_x^2}\varphi_x, \tag{9}$$

$$\varphi_x = \begin{cases} k_x \operatorname{ch} p_x \bar{x}, \quad k_x = \dfrac{2\operatorname{sh} p_x \Delta\bar{\xi} \operatorname{ch} p_x (1-\bar{\xi})}{\operatorname{sh} p_x}; \quad \bar{x} \in [0;\ \bar{\xi} - \Delta\bar{\xi}], \\ k_x \operatorname{ch} p_x \bar{x} - \operatorname{ch} p_x (\bar{x} - \bar{\xi} + \Delta\bar{\xi}) + 1; \quad \bar{x} \in [\bar{\xi} - \Delta\bar{\xi};\ \bar{\xi} + \Delta\bar{\xi}], \\ k_x \operatorname{ch} p_x \bar{x} - \operatorname{ch} p_x (\bar{x} - \bar{\xi} + \Delta\bar{\xi}) + \operatorname{ch} p_x (\bar{x} - \bar{\xi} - \Delta\bar{\xi}); \\ \bar{x} \in [\bar{\xi} + \Delta\bar{\xi};\ 1]. \end{cases}$$

Following [3], we shall seek an approximate solution in the form

$$\tilde{N} = \langle N_x \rangle N_y. \tag{10}$$

The solution of the boundary-value problem (4), (5) is equivalent to finding the minimum of the functional

$$J(N) = \int_0^1\int_0^1 \left[\varepsilon_x \left(\frac{\partial N}{\partial \bar{x}}\right)^2 + \varepsilon_y \left(\frac{\partial N}{\partial \bar{y}}\right)^2 + b^2N^2 - 2N1\{U\}\right] d\bar{x}\,d\bar{y}. \tag{11}$$

Substituting $\tilde{N}$ in the form (10) into (11) and carrying out integration with respect to the variable $\bar{x}$, we obtain the problem of the minimum of a simple integral,

$$J(N_y) = \int_0^1 \left[b_1 \left(\frac{dN_y}{d\bar{y}}\right)^2 + b_2 N_y^2 - 1\{\bar{y}\}\, 2b_3 N_y\right] d\bar{y},$$

$$b_1 = \frac{1}{p_x}\left[\frac{k_x^2}{4}\operatorname{sh} 2p_x + \frac{1}{2}\operatorname{sh} 2p_x(1-\bar{\xi})\operatorname{ch} 2p_x\Delta\bar{\xi} - 2k_x \operatorname{ch} p_x \operatorname{sh} p_x\Delta\bar{\xi}\operatorname{ch} p_x(1-\bar{\xi}) - \operatorname{sh} p_x r_2 \operatorname{ch} p_x r_1\right] +$$

$$+ k_x r_1 \operatorname{ch} p_x(\bar{\xi} - \Delta\bar{\xi}) - k_x r_2 \operatorname{ch} p_x(\bar{\xi} + \Delta\bar{\xi}) - \frac{1}{2}k_x^2 + r_2 \operatorname{ch} 2p_x\Delta\bar{\xi} + \bar{\xi} - 1, \tag{12}$$

$$b^2 = \left(\frac{2\Delta\eta}{\varepsilon_x p_x^2}\right)^2 \left\{\frac{k_x^2}{4}\operatorname{sh} 2p_x + \frac{1}{2p_x}\operatorname{sh} 2p_x(1-\bar{\xi})\operatorname{ch} 2p_x\Delta\bar{\xi} - 4\operatorname{sh} p_x(1-\bar{\xi})\operatorname{ch} p_x\Delta\bar{\xi} + 3(1-\bar{\xi}) + \frac{1}{2}k_x^2 -\right.$$

$$-k_x\left[\frac{2}{p_x}\operatorname{ch} p_x \operatorname{sh} p_x \Delta\bar{\xi}\operatorname{ch} p_x(1-\bar{\xi}) + \frac{4}{p_x}\operatorname{sh} p_x(1-\operatorname{ch} p_x\Delta\bar{\xi}) + r_1 \operatorname{ch} p_x(\bar{\xi}-\Delta\bar{\xi}) - r_2 \operatorname{ch} p_x(\bar{\xi}+\Delta\bar{\xi})\right] -$$

$$\left. - \frac{1}{p_x}\operatorname{sh} p_x r_2 \operatorname{ch} p_x r_1 + \frac{8}{p_x}\operatorname{ch} p_x\Delta\bar{\xi}\operatorname{sh} p_x\frac{r_2}{2}\operatorname{ch} p_x\frac{r_1}{2} - r_2(\operatorname{ch} 2p_x\Delta\bar{\xi} + 2)\right\},$$

$$b_3 = \frac{2\Delta\bar{\eta}}{\varepsilon_x p_x^2}\left\{\frac{1}{p_x}\left[2\operatorname{sh} p_x\Delta\bar{\xi}(k_x \operatorname{ch} p_x\bar{\xi} - \operatorname{ch} p_x\Delta\bar{\xi}) + 2\Delta\bar{\xi}\right]\right\},$$

$$r_1 = 1 - \bar{\xi} + \Delta\bar{\xi}, \quad r_2 = 1 - \bar{\xi} - \Delta\bar{\xi}.$$

The function N_y, realizing the minimum of the functional (11), is the solution of the boundary-value problem for the equation

$$\frac{d^2N_y}{d\bar{y}^2} - p_y^2 N_y = -1\{\bar{y}\}\frac{b_3}{\varepsilon_y b_2}, \tag{13}$$

$$p_y^2 = \frac{\varepsilon_x b_1 + b^2 b_2}{\varepsilon_y b_2}. \tag{14}$$

Solving Eq. (13) with the boundary conditions

$$\left.\frac{dN_y}{d\bar{y}}\right|_{\bar{y}=0} = \left.\frac{dN_y}{d\bar{y}}\right|_{\bar{y}=1} = 0, \tag{15}$$

we find the solution for N_y analogously to N_x:

$$\langle N_y \rangle = \frac{b_3}{\varepsilon_x b_1 + b^2 b_2}\varphi_y, \tag{16}$$

$$\varphi_y = \begin{cases} k_y \operatorname{ch} p_y \bar{y}, \; k_y = \dfrac{2 \operatorname{sh} p_y \Delta\bar{\eta} \cdot \operatorname{ch} p_y (1-\bar{\eta})}{\operatorname{sh} p_y}, & \bar{y} \in [0, \; \bar{\eta} - \Delta\bar{\eta}], \\ k_y \operatorname{ch} p_y \bar{y} - \operatorname{ch} p_y (\bar{y} - \bar{\eta} + \Delta\bar{\eta}) + 1, & \bar{y} \in [\bar{\eta} - \Delta\bar{\eta}; \; \bar{\eta} + \Delta\bar{\eta}], \\ k_y \operatorname{ch} p_y \bar{y} - \operatorname{ch} p_y (\bar{y} - \bar{\eta} + \Delta\bar{\eta}) + \operatorname{ch} p_y (\bar{y} - \bar{\eta} - \Delta\bar{\eta}), & \bar{y} \in [\bar{\eta} + \Delta\bar{\eta}; \; 1]. \end{cases} \tag{17}$$

The relations (9), (10), (16) allow us to write the approximate solution of the problem (4), (6) being considered

$$\tilde{N} = N_0 (k_x \operatorname{ch} p_x \bar{x} - \varphi_{2x} + \varphi_{3x})(k_y \operatorname{ch} p_y \bar{y} - \varphi_{2y} + \varphi_{3y}), \tag{18}$$

$$N_0 = \frac{2\Delta\bar{\eta} b_3}{\varepsilon_x p_x^2 (\varepsilon_x b_1 + b^2 b_2)}. \tag{19}$$

The value of the temperature of the middle surface is found from the definition

$$\tilde{N}_s = \int_0^1 \int_0^1 \tilde{N} d\bar{x} d\bar{y}. \tag{20}$$

Substituting into (20) the value $\tilde{N}$ from (18) and carrying out the integration, we obtain

$$\tilde{N}_s = N_0 \; 4\Delta\bar{\xi}\Delta\bar{\eta}. \tag{21}$$

On the other hand, the mean temperature of the plate, $\bar{\vartheta}_s$, is connected with the power P of the source by the relation

$$\bar{\vartheta}_s = \frac{P}{(\alpha_1 + \alpha_2) L_x L_y} \tag{22}$$

or in the dimensionless form

$$N_s = \frac{4\Delta\bar{\xi}\Delta\bar{\eta}}{b^2}, \text{ where } N_s = \frac{4\Delta\xi\Delta\eta\bar{\vartheta}_s\lambda}{P\delta}. \tag{23}$$

From the comparison of the expressions (21) and (23) we obtain a simple expression for calculating the constant coefficient N_0:

$$N_0 = 1/b^2. \tag{24}$$

It can be shown that the quantity p_y^2 weakly depends on the coordinate $\bar{\xi}$ and the dimension $\Delta\bar{\xi}$, and for $\bar{\xi} \to 1$ the expression (14) for p_y^2 is simplified and assumes the form

$$p_y^2 = \frac{2b^2}{\varepsilon_y}\left(1 - \frac{1}{\dfrac{1}{2p_x} \operatorname{sh} 2p_x + 1}\right). \tag{25}$$

TABLE 1. Calculation Results

$\bar{\eta}$	$\Delta\bar{\xi}/\Delta\bar{\eta}$	$B=\frac{\alpha L_x^2}{\lambda\delta}$	$L_x/L_y=0,5$ $L_x=0,2$ $L_y=0,4$	1 $L_x=0,2$ $L_y=0,2$	2 $L_x=0,2$ $L_y=0,1$
	0,5	54	−11,9	−10,6	−9,00
	$\Delta\bar{\xi}=0,1$	13,5	−16,0	−11,6	−3,9
	$\Delta\bar{\eta}=0,2$	2,7	−11,1	− 4,4	−0,6
	1	54	−10,9	− 9,4	−7,0
$\Delta\bar{\eta}$	$\Delta\bar{\xi}=0,1$	13,5	−12,5	− 7,4	2,3
	$\Delta\bar{\eta}=0,1$	2,7	− 6,6	1,7	5,2
	2	54	− 9,4	− 7,0	−4,1
	$\Delta\bar{\xi}=0,1$	13,5	− 7,0	− 1,1	7,3
	$\Delta\bar{\eta}=0,05$	2,7	0,9	7,4	8,5
	0,5	54	− 5,2	−10,0	−8,8
	$\Delta\bar{\xi}=0,1$	13,5	−14,4	−10,8	−7,6
	$\Delta\bar{\eta}=0,2$	2,7	− 8,8	− 4,6	−1,3
	1	54	−11,7	− 6,4	−5,2
0,25	$\Delta\bar{\xi}=0,1$	13,5	− 7,4	− 3,2	0,6
	$\Delta\bar{\eta}=0,1$	2,7	− 2,2	1,4	0,1
	2	54	− 6,9	− 3,2	−2,5
	$\Delta\bar{\xi}=0,1$	13,5	0,2	2,7	3,7
	$\Delta\bar{\eta}=0,05$	2,7	5,2	5,8	3,0
	0,5	54	− 4,6	− 1,1	−6,2
	$\Delta\bar{\xi}=0,1$	13,5	−14,0	− 7,2	−2,2
	$\Delta\bar{\eta}=0,2$	2,7	− 6,7	− 2,5	−2,8
	1	54	−10,4	− 6,5	−3,1
0,5	$\Delta\bar{\xi}=0,1$	13,5	− 7,5	0,2	3,2
	$\Delta\bar{\eta}=0,1$	2,7	0,8	4,1	0,4
	2	54	− 6,5	− 3,2	−7,8
	$\Delta\bar{\xi}=0,1$	13,5	0,0	5,4	6,6
	$\Delta\bar{\eta}=0,05$	2,7	8,1	8,5	2,4

A subsequent analysis allows us to recommend the following expressions for determining p_x, p_y:

$$p_x=\sqrt{\frac{b^2}{\varepsilon_x}\left[1.5-\left(\frac{\operatorname{sh} 2\sqrt{\frac{b^2}{\varepsilon_y}}}{2\sqrt{\frac{b^2}{\varepsilon_y}}}+1\right)^{-1}\right]}, \tag{26}$$

$$p_y=\sqrt{\frac{b^2}{\varepsilon_y}\left[1.5-\left(\frac{\operatorname{sh} 2\sqrt{\frac{b^2}{\varepsilon_x}}}{2\sqrt{\frac{b^2}{\varepsilon_x}}}+1\right)^{-1}\right]}. \tag{27}$$

Taking into account the notation (3), from (18) and (24) the approximate solution for the temperature ϑ with dimensions can be written as follows:

$$\vartheta=\frac{P}{\alpha S_{\text{source}}}\varphi_x\varphi_y,\ \alpha=\alpha_1+\alpha_2, \tag{28}$$

$S_{\text{source}} = 4\Delta\xi\Delta\eta$ is the area occupied by the source.

To determine the error of the solution (28), we compared the results of the exact solution ϑ and the approximate solution $\tilde{\vartheta}$ in a wide range of variation of parameters.

The results of the calculations are summarized in Table 1, where we have presented the error $\delta\vartheta$ of the approximate solution for the central point of the source, calculated from the expression

$$\delta\vartheta=\frac{\vartheta-\tilde{\vartheta}}{\vartheta}\cdot 100\%.$$

We note that different cases of variation of all parameters are considered: plate dimensions L_x, L_y (L_x = 0.2 m, L_y = 0.4; 0.2; 0.1) and L_x/L_y = 0.5; 1; 2; source dimensions $\Delta\xi$, $\Delta\eta$ ($\Delta\xi$ = 0.1; $\Delta\eta$ = 0.2; 0.1; 0.5) and $\Delta\xi/\Delta\eta$, ($\Delta\xi/\Delta\eta$ = 0.5; 1; 2); coordinates of the place of location of the source ξ = 0.5; η (η = 0.5; 0.25; 0.2; 0.1; 0.05); the Biot criterion ($Bi = \alpha L_x^2/\lambda\delta$ = 54; 13; 2.7). The mean-square error equals 6.6%.

NOTATION

t, local temperature of the plate at the point with coordinates x, y; t_c, temperature of the surrounding medium, °K; L_x, L_y, plate dimensions, m; $2\Delta\xi$; $2\Delta\eta$, dimensions of the source region, m; ξ, η, coordinate of the center of the source zone, m; λ, coefficient of thermal conductivity, W/mK; α_1, α_2, heat-exchange coefficients of the plate surfaces, W/m²/°K; P, heat flux dissipated in the plate, W; δ, plate thickness, m.

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